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Quantum kinematics and the Lie group structure of non-Abelian quantum mechanics

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Abstract. A formalism of quantum kinematics which rests on the regular representation of a Lie group is proposed. This representation leads to explicit canonical commutation relations for non-Abelian dynamical variables which, together with the Lie algebra, define the kinematic algebra. Generalised equations of motion for the group-parameter-dependent operators, as well as generalised wave equations, are introduced over the group manifold considered as a background arena. The formalism proposed affords a method of geometric quantisation stemming directly from the observed symmetries of a system.

The recent trend in elementary particle theories, where several SU(n) and other gauge groups play a fundamental role (Wilson 1974), sets the problem of quantising a system which is primarily described by non-Abelian dynamical variables (Yamada 1982). In general this seems to be an unsolved problem of quantum kinematics because the conventional canonical commutation relations (i.e., cf equation (1) below) apply only to Cartesian coordinates and their conjugate Abelian momenta and not to general dynamical variables, and also because Weyl's (1931) quantisation is not flexible enough for the purposes of physics (Daubechies 1983) since it contains the non-Abelian fundamental commutators only implicitly. The same problem also arises in connection with some recent attempts to generalise the theory of minimum-uncertainty states (Yamada 1982) to variables other than Cartesian (Carruthers and Nieto 1968, Nieto 1967, Nieto *et al* 1981 and references quoted therein). The quantum kinematics of non-Abelian variables are also an indispensable device in the formulation of lattice gauge theories which circumvent conventional perturbation theory (Wilson 1974) in the search for the appropriate description of quark confinement (Kogut and Susskind 1975, Creutz 1977, Kogut 1980).

Quantum kinematics were successfully initiated by Weyl (1931) some fifty years ago. Weyl's most interesting achievement in this respect was his discussion of Heisenberg's kinematics as an Abelian group of unitary transformations. Indeed, Weyl's deduction of the fundamental commutation rule

$$[Q^a, P_b] = i\hbar\delta^a_b \tag{1}$$

(from the assumed space translation symmetry) still remains as almost the unique contribution of quantum kinematics which rests on firm ground, apart from the relevant Lie algebras and their general representations.

This paper is a brief report of work in progress concerning non-Abelian quantum kinematics. Although our results are not exemplified in this paper and are thus purely formal (in fact, they belong to the general formalism of quantum mechanics), we hope that they may be of some interest for people working in the foundations and possible generalisations of quantum theory. Here we present some features of a general formalism of quantum kinematics which one obtains rather naturally when one considers the regular representation of a Lie group, whether Abelian or not. In this paper we show how a generalised wave mechanics may be defined over the group manifold itself. In particular, our approach may be a way out of the problem mentioned above since we obtain non-Abelian canonical commutators (cf equation (16) below) which can be evaluated systematically (by purely group-theoretical methods) for any given Lie group. The generalised commutation relations arise from the global symmetries of the system (in contrast to the Lie algebra commutators which arise only locally) and, moreover, they are not necessarily committed to (nor do they presuppose) a canonical formalism attached to any sort of classical analogue. Clearly, for an Abelian group (as the group of space translations) equation (16) yields equation (1). This fact indeed represents the main motivation of the general approach to quantum kinematics as sketched in this paper. One expects that equation (16) will also set the general framework for having *non-Abelian quantum mechanics*, that is, for having a quantisation scheme resting on fundamental commutation relations for dynamical variables of any kind (not only Cartesian and Abelian).

Let G denote an r-parameter Lie group which in some realisation is characterised by a set of r group multiplication functions (cf, e.g., Racah 1965): $g^a(q'^1, \ldots, q'^r; q^1, \ldots, q^r) = q''^a$, say, with $a = 1, \ldots, r$, and where the q correspond to essential parameters of G. We will denote by $q = (q^1, \ldots, q^r)$ a generic point of the group manifold $M(\mathrm{G})$. For the sake of simplicity and in order to concentrate our attention on the main ideas, we handle only the identity component $\mathrm{G_e}$ of G (i.e., $\mathrm{G} = \mathrm{G_e}$ henceforth). Then we define the *regular representation* $U(\mathrm{G})$ of G by means of continuous matrices $U(q)$, $q \in M(\mathrm{G})$, with matrix elements given by

$$U_{q'q''}(q) = \mu_0^{-1} L(q') \delta^{(r)}(q' - g(q; q'')) \tag{2}$$

where μ_0 is an arbitrary constant, q', q'' are also points in $M(\mathrm{G})$, $\delta^{(r)}(q)$ denotes the r-fold Dirac delta and $L(q')$ is the $r \times r$ determinant of

$$L^a_b(q') = \lim_{q \to \mathrm{e}} \partial_b g^a(q'; q), \tag{3}$$

e being the 'identity point' in $M(\mathrm{G})$. In effect, it can be easily shown that

$$\int \mathrm{d}\mu_{\mathrm{L}}(q'') U_{q'q''}(q_1) U_{q''q'''}(q_2) = U_{q'q'''}(g(q_1; q_2)) \tag{4}$$

where $\mathrm{d}\mu_{\mathrm{L}}(q) = \mu_0 L^{-1}(q)\, \mathrm{d}q^1 \ldots \mathrm{d}q^r$ is the Hurwitz left-invariant measure of G (Weyl 1931, Wigner 1959).

Next, let $H(\mathrm{G})$ be the Hilbert space which carries the (left) regular representation of G, and let $\{|q\rangle\}$ be a continuous complete basis of $H(\mathrm{G})$ such that

$$U_{q'q''}(q) = \langle q'|U(q)|q''\rangle \tag{5}$$

where, clearly, $U(q)$ are the linear operators of this representation. Thus one has

$$\langle q|q'\rangle = \mu_0^{-1} L(q) \delta^{(r)}(q - q') \tag{6}$$

$$\int \mathrm{d}\mu_{\mathrm{L}}(q) |q\rangle\langle q| = I \tag{7}$$

where I denotes the identity operator in $H(\mathrm{G})$. Furthermore, one also obtains

$$U(q)|q'\rangle = |g(q;q')\rangle \tag{8}$$

and, in particular,

$$U(q)|\mathrm{e}\rangle = |q\rangle. \tag{9}$$

Equation (8) shows neatly that we are handling the left regular representation of G. Finally, the identity

$$L(q'')\delta^{(r)}(q''-g(q;q')) = L(q')\delta^{(r)}(q'-g(\bar{q};q'')) \tag{10}$$

enables one to show that the operators $U(q)$ of the regular representation are unitary:

$$U^{+}(q) = U^{-1}(q) = U(\bar{q}). \tag{11}$$

Here we denote by $\bar{q}$ that point in $M(\mathrm{G})$, uniquely associated with the point $q \in M(\mathrm{G})$, such that $g(q;\bar{q}) = g(\bar{q};q) = \mathrm{e}$ (Racah 1965).

After these prolegomena we are ready to consider the quantum kinematics associated with the regular representation $U(\mathrm{G})$. First we formally introduce generalised 'position' operators on the group manifold by means of the following spectral representation:

$$Q^{a} = \int \mathrm{d}\mu_{\mathrm{L}}(q)|q\rangle q^{a}\langle q|. \tag{12}$$

These are commuting Hermitian operators such that $Q^{a}|q\rangle = q^{a}|q\rangle$ and so they provide us with a complete set of compatible observables in $H(\mathrm{G})$. Of course, we also consider the infinitesimal generators P_a of $U(\mathrm{G})$ defined by $U(\mathrm{e}+\delta q) = I - (\mathrm{i}/\hbar)\delta q^{a}P_{a}$ as usual. These are Hermitian operators obeying the Lie algebra associated with $U(\mathrm{G})$, say

$$[P_a, P_b] = \mathrm{i}\hbar f^{c}_{ab}P_c, \tag{13}$$

and which play the role of generalised 'momentum' operators on $M(\mathrm{G})$. The Lie algebra is the only piece of information one usually takes into account in the current approach to non-Abelian quantum kinematics. The formalism, however, is much richer than this, since the set $\{Q^1, \ldots, Q^r; P_1, \ldots, P_r\}$, and *not* just $\{P_1, \ldots, P_r\}$, is the *irreducible set of observables* which defines the quantum model associated with G.

Indeed, if we study the kinematics of the operators Q^a, that is, the (active) transformation law

$$\hat{Q}^{a}(q) = U^{+}(q)Q^{a}U(q), \tag{14}$$

we find

$$\hat{Q}^{a}(q) = \int \mathrm{d}\mu_{\mathrm{L}}(q')|q'\rangle g^{a}(q;q')\langle q'| = g^{a}(q;Q). \tag{15}$$

Therefore a straightforward calculation yields the generalised commutation rule

$$[Q^{a}, P_b] = \mathrm{i}\hbar R^{a}_{b}(Q) \tag{16}$$

where the Hermitian operators $R^{a}_{b}(Q)$, $a, b = 1, \ldots, r$, have the spectral representation

$$R^{a}_{b}(Q) = \int \mathrm{d}\mu_{\mathrm{L}}(q)|q\rangle R^{a}_{b}(q)\langle q| \tag{17}$$

with

$$R^a_b(q) = \lim_{q' \to e} \partial'_b g^a(q'; q). \tag{18}$$

The $R^a_b(q)$ are the elements of the right-transport matrix for contravariant vectors from the identity point e to the point q in the global affine space $M(G)$. (In the same manner, the $L^a_b(q)$ defined in equation (3) correspond to left transport.) In this way we also get (cf equation (8))

$$P_a|q\rangle = i\hbar R^b_a(q)\partial_b|q\rangle. \tag{19}$$

This result is interesting for it generalises the usual (i.e., Abelian) Fourier mapping $P_a \to -i\hbar\partial_a$ to the non-Abelian case: $P_a \to -i\hbar R^b_a(q)\partial_b$. Once the group multiplication functions $g^a(q'; q)$ for the essential parameters of a Lie group G are known, the canonical commutators stated in equation (16) can be evaluated easily. Of course, for an Abelian Lie group one has $R^a_b(q) = L^a_b(q) = \delta^a_b$ and thus equation (16) becomes equation (1) as required.

For the kinematics of the generalised 'momentum' operators, namely

$$\hat{P}_a(q) = U^\dagger(q) P_a U(q), \tag{20}$$

one gets

$$\hat{P}_a(q) = R^b_a(q)\bar{L}^c_b(q)P_c \tag{21}$$

with $\bar{L}^b_a(q)L^c_b(q) = \delta^c_a$ (from which the Lie algebra follows).

Finally, we observe that the operators Q^a and P_a are completely independent of q; in this sense, they belong to the 'Schrödinger picture' of the kinematics. On the other hand, the q-dependent operators $\hat{Q}^a(q)$ and $\hat{P}_a(q)$ belong to the 'Heisenberg picture'. Furthermore, it can be shown that

$$R^c_b(q)\partial_c\hat{Q}^a(q) = (i/\hbar)[\hat{P}_b(q), \hat{Q}^a(q)] \tag{22}$$

$$R^c_b(q)\partial_c\hat{P}_a(q) = (i/\hbar)[\hat{P}_b(q), \hat{P}_a(q)]. \tag{23}$$

We interpret these equations as the generalised equations of 'motion' of the q-dependent operators with respect to the parameter space $M(G)$ considered as a background arena. In the same way, we interpret the equations

$$R^b_a(q)\partial_b\langle q|\psi\rangle = (i/\hbar)\langle q|P_a|\psi\rangle \tag{24}$$

as the generalised 'wave equations' of quantum kinematics on $M(G)$.

We end this paper with some remarks. First we observe that quantum kinematics afford a theoretical basis for having non-Abelian 'wave mechanics' over the group manifold (Q representation). Equation (19) (cf equation (24)) completes the Q representation while equation (16) states the fundamental commutators of the regular $U(G)$ kinematics. Equations (22) and (23) are completely analogous to the Heisenberg equations of motion of the ordinary theory and represent a direct generalisation of it. In the same spirit, equations (24) are a direct generalisation of Schrödinger's time-dependent equation. It is important to remark, however, that in quantum kinematics one treats all variables (i.e., all group parameters) on the same footing and therefore time is not a preferred parameter.

One should not misinterpret the possible physical meaning of the kinematics, since in the applications it may be intimately related to quantum dynamics. In effect, it is clear that for any well defined isolated system the group G must include *all* the (external and internal) symmetries of the system. But then, in flat spacetime theories for instance, the Poincaré group must be a subgroup of G and thus the kinematics distinguish the Hamiltonian operator from the other Q and P operators as the generator of time translation invariance. Plainly so. The main point, however, is that the Hamiltonian appears to be related automatically to the other operators of the kinematics by means of the set of commutators (the kinematic algebra) arising not only from the Lie algebra but also from the global affine structure of the group manifold. Indeed, it is the whole symmetry group G, not only the Hamiltonian, that operates as the basic ingredient of the outcoming quantum model. Therefore it seems worthwhile to examine whether the kinematic equations of motion (cf equations (22), (23) and (24)) have some dynamical meaning, at least for those physical systems which manifest empirically the symmetries inherent to the adopted model group G. This approach may be especially useful for high-energy physics where classical analogues are missing or rather difficult to guess. Moreover, a closer investigation regarding the dynamical contents of symmetry groups in mechanics (Mariwalla 1975, Aguirre and Krause 1984a, b) would be desirable.

In summary, it seems possible to reinterpret quantum kinematics as a programme of geometric quantisation stemming directly from the observed symmetries of a system. Although there are several contributions following this idea in the current literature (Aldaya and Azcárraga 1982, Prosser 1983 and references therein), this author has been unable to find a discussion of non-Abelian kinematics as introduced in this paper. In particular, the formalism proposed includes (and looks simpler than) Weyl's quantisation approach and also differs substantially from the traditional methods of geometric quantisation of Souriau (1970), Kostant (1970) and others (Simms and Woodhouse 1976).

Details and applications of non-Abelian quantum kinematics will be published elsewhere.

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